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Nontrivial Fixed Points and Screening in the Hierarchical Two-Dimensional Coulomb Gas

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We show the existence and asymptotic stability of two fixed points of the renormalization group transformation for the hierarchical two-dimensional Coulomb gas in the sine-Gordon representation and temperatures slightly greater than the critical one. We prove also that the correlations at the fixed points decay as in the hierarchical massive scalar free theory, that is, as d_{xy}^{-4}. We argue that this is the natural definition of screening in the hierarchical approximation.

KEY WORDS: Coulomb systems; renormalization group.

1. INTRODUCTION

In the last 10 years the renormalization group ideas have been extensively applied to the two-dimensional Coulomb gas of identical charges $\pm e$, in order to rigorously understand to so-called Kosterlitz–Thouless phase transition.[1]

For inverse temperatures β larger than the critical one β_c and small activity λ, it has been proved that there is no screening.[2,3] This result is strictly related to the fact that, in the field-theoretic representation of the model (the so-called sine-Gordon representation), the effective potential goes to zero as the scale goes to infinity.[4] All these results are valid also in the hierarchical approximation of the model (see Section 2), where they are obtained much more easily.[5,6]

For $\beta < \beta_c$ screening is generally expected to be found, but this property has been proved in the exact model only for β very small.[7]

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However, in the case of the hierarchical approximation, a weak form of screening has been proved for $\beta \lesssim \beta_c$.[6]

In this paper we study the hierarchical Coulomb gas in the sine-Gordon representation in the region $\beta \lesssim \beta_c$, by analyzing the renormalization group transformation $\mathbf{T}$ of the effective potential. We prove that $\mathbf{T}$ has two nontrivial asymptotic stable fixed points, which have the following screening property: the two-charge truncated correlations decay as d_{xy}^{-4} as the (hierarchical) distance d_{xy} goes to infinity, which is the decaying behavior of the truncated correlations of the massive hierarchical scalar field. Our analysis also suggests that the effective potential for the hierarchical Coulomb gas on scale 1 is in the domain of attraction of one of the fixed points, if the activity is small enough, so that in this case as well screening should be observed.

The existence of the nontrivial fixed points was proved in ref. 6 with a different technique, which does not use the sine-Gordon representation and extends to more general models. However, the analysis of ref. 6 allows one to study only the simple (not truncated) correlations. This is why it was restricted to the screening for fractional charges; in fact, for the fractional charges, the truncated correlations coincide with the simple ones.

Finally, we want to stress that the technique used in this paper is essentially based on the bound discussed in the Appendix, which is a bound for the Ursell coefficients of a system of arbitrary charges sitting in the same point and interacting with a potential cQ_iQ_j. We were unable to find this estimate, which we think is interesting by itself, in the literature.

2. THE HIERARCHICAL MODEL

Let Q_j, $j \in \mathbb{N}$, be a sequence of compatible pavements of $\mathbb{R}^2$ made of squares of side size γ^j, where $\gamma \geqslant 2$ is an integer. To each $\Delta \in Q_j$ we associate a Gaussian variable z_Δ such that

$$\mathscr{E}(z_\Delta^2) = \frac{1}{2\pi} \log \gamma, \qquad \mathscr{E}(z_\Delta z_{\Delta'}) = 0 \quad \text{if} \quad \Delta \neq \Delta' \tag{2.1}$$

Then we define, $\forall x \in \mathbb{R}^2$,

$$\varphi_x = \sum_{k=0}^{\infty} z_{\Delta_x^{(k)}} \tag{2.2}$$

where $\Delta_x^{(k)}$ is the tessera of side size γ^k containing x.

Given $x, y \in \mathbb{R}^2$, let h_{xy} be the smallest integer such that exists a

$\Delta \in Q_{h_{xy}}$ containing both x and y. We shall call $d_{xy} \equiv \gamma^{h_{xy}}$, the size of Δ, the hierarchical distance between x and y. By using (2.1), it is easy to see that

$$\mathscr{E}((\varphi_x - \varphi_y)^2) = \mathscr{E}\left(\sum_{k=0}^{h_{xy}-1} [z_{\Delta_x^{(k)}} - z_{\Delta_y^{(k)}}]^2\right) = \frac{1}{\pi} \log d_{xy} \tag{2.3}$$

which justifies the claim that φ_x is a reasonable approximation of the two-dimensional zero-mass scalar field. In the following we shall denote the corresponding Gaussian measure by $P(d\varphi)$.

If $v(z)$, $z \in \mathbb{R}^1$, is a real function such that

$$v(0) = 0, \qquad v(z) = v(-z) \tag{2.4}$$

and Λ is a finite volume belonging to Q_R, for some $R > 0$, we shall consider the measure

$$\mu_v^\Lambda(d\varphi) = \frac{1}{Z_v^\Lambda} P(d\varphi) \prod_{\Delta \in Q_0 \cap \Lambda} e^{v(\varphi_\Delta)} \tag{2.5}$$

$$Z_v^\Lambda = \int P(d\varphi) \prod_{\Delta \in Q_0 \cap \Lambda} e^{v(\varphi_\Delta)} \tag{2.6}$$

where φ_Δ is the constant value of the field on the tessera Δ.

The choice

$$v(\varphi) = \lambda[\cos(\alpha\varphi) - 1] \tag{2.7}$$

corresponds to the hierarchical Coulomb gas in the volume Λ with activity $\lambda/2$, charges $\pm e$, and temperature β^{-1}, such that

$$\beta e^2 = \alpha^2 \tag{2.8}$$

For more details on this point see ref. 5, where a rescaled field was used, instead of (2.2).

Another interesting choice is

$$v(\varphi) = -m^2\varphi^2 \equiv u_m(\varphi) \tag{2.9}$$

which should give rise to the two-dimensional hierarchical scalar field of mass m. It is important to remark, however, that this is not a good approximation of the massive scalar field. In fact, it is easy to show that

$$\lim_{\Lambda \to \mathbb{R}^2} \int \mu_{u_m}^\Lambda(d\varphi)\, \varphi_x \varphi_y \propto d_{xy}^{-4} \tag{2.10}$$

in disagreement with the exponential decay of the massive scalar field (a similar property is valid in other hierarchical models; see ref. 8, Chapter 4, Exercise 2).

This observation will be very relevant in the following, since it implies that the hierarchical Coulomb gas should have power-decaying correlations also in a screened phase, but with a power equal to 4 independently of β.

Let us now define the renormalization group transformation.

If $F(z)$ is a function on $\mathbb{R}$ and $\langle \cdot \rangle_v^\Lambda$ denotes the expectation w.r.t. the measure (2.5), then

$$\langle F(\varphi_0)\rangle_v \equiv \lim_{\Lambda \to \mathbb{R}^2} \langle F(\varphi_0)\rangle_v^\Lambda = \langle \mathbf{L}_{\mathbf{T}^{k-1}v} \cdots \mathbf{L}_v F(\varphi_0)\rangle_{\mathbf{T}^k v} \tag{2.11}$$

where

$$(\mathbf{T}v)(\varphi) = \log\left[\frac{\int P_0(dz)\, e^{v(\varphi+z)}}{\int P_0(dz)\, e^{v(z)}}\right]^{\gamma^2} \tag{2.12}$$

$$(\mathbf{L}_v F)(\varphi) = \frac{\int P_0(dz)\, e^{v(\varphi+z)}\, F(\varphi+z)}{\int P_0(dz)\, e^{v(\varphi+z)}} \tag{2.13}$$

if $P_0(dz)$ denotes the Gaussian measure on $\mathbb{R}^1$ of mean zero and covariance $(1/2\pi)\log\gamma$.

The operators $\mathbf{T}$ and $\mathbf{L}_v$ appear also in the expressions similar to (2.11) valid for the expectations of any *observable* depending on the values of the field φ_x in a finite set of tesserae $\Delta \in Q_0$.

The operator (2.12) is the *renormalization group transformation*. It leaves invariant the space $\mathfrak{C}_\alpha$ of the continuous functions periodic of period $T_\alpha = 2\pi/\alpha$ and satisfying (2.4); then, if we want to study the hierarchical Coulomb gas at temperature β^{-1}, we have to restrict $\mathbf{T}$ to $\mathfrak{C}_\alpha$ with $\alpha = (\beta e^2)^{1/2}$. In ref. 5 it was implicitly shown that $v_0(\varphi) = 0$ is, for $\alpha^2 > 8\pi$, a fixed point of (2.12), which is attracting for functions of the form (2.7), for λ small enough. In fact, one could also show that it is locally attracting in some subspace of sufficiently regular functions.

In this paper we shall study the more difficult case $\alpha^2 \leqslant 8\pi$ and we shall prove that there are two stable fixed points $v_\alpha(\varphi) \neq 0$ (this result, as discussed in the Introduction, has already been obtained with a different technique[6]) is a suitable subspace of $\mathfrak{C}_\alpha$.

Moreover, by studying the spectrum of (2.13) for $v = v_\alpha$ in the space of L_2 functions periodic of period T_α, we shall prove that the integer charge truncated correlations decay like d_{xy}^{-4}. The restriction of $\mathbf{L}_v$ to periodic functions is motivated by the fact that the truncated integer charge correlations are given by the formula

$$\rho^T(x_1, \sigma_1;...; x_n, \sigma_n) = \frac{\partial^n}{\partial\omega_1 \cdots \partial\omega_n} \log \left\langle \exp\left\{\frac{\lambda}{2} \sum_{j=1}^{n} \omega_j \exp(i\alpha\sigma_j\varphi_j)\right\}\right\rangle\Big|_{\omega_j=0} \tag{2.14}$$

where $\sigma_i \in \{-1, 1\}$ are the charges and x_i are the positions of n particles.

3. THE EXISTENCE OF TWO ATTRACTING NONTRIVIAL FIXED POINTS

The proof of the existence of nontrivial fixed points for $\alpha^2 < 8\pi$ is based on the perturbative expansion of the renormalization group transformation (2.12). If $v(\varphi) \in \mathfrak{C}_\alpha$, we can write

$$v(\varphi) = \sum_{0 \neq Q \in \mathbb{Z}} v_Q(e^{i\alpha Q\varphi} - 1) \tag{3.1}$$

where $v_Q = v_{-Q}$. Then, if v'_Q are the Fourier coefficients of $(\mathbf{T}v)(\varphi)$, (2.12) can be written in the following form:

$$v'_Q = \gamma^{2-(\alpha^2/4\pi)Q^2} v_Q + \gamma^2 \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{Q_1 + \cdots + Q_n = Q} v_{Q_1} \cdots v_{Q_n} F_n(Q_1,..., Q_n) \tag{3.2}$$

where

$$F_n(Q_1,..., Q_n) = \mathscr{E}^T(e^{i\alpha Q_1 z};...; e^{i\alpha Q_n z}) \tag{3.3}$$

if we denote $\mathscr{E}^T$ the truncated expectation with respect to the measure $P_0(dz)$.

In the Appendix we prove the following nontrivial bound, which will play a crucial role in the following:

$$|F_n(Q_1,..., Q_n)| \leqslant n^{n-1} \left[\prod_{i=1}^{n} (4\sqrt{\kappa}\, |Q_i|)\right] e^{-\kappa Q^2/n} \tag{3.4}$$

where $\kappa = (\alpha^2/8\pi) \log \gamma$ and $Q = \sum_{i=1}^{n} Q_i$.

The linearization of (3.2) around the fixed point $v = 0$ has the property that the Fourier coefficients $v_{\pm 1}$ become unstable at $\alpha^2 = 8\pi$. This implies, as is well known, a bifurcation of the trivial fixed point, which is responsible for the Kosterlitz–Thouless phase transition. In this paper we study the range of temperatures immediately above the critical one, given by the relation

$$0<\varepsilon\equiv 2-\frac{\alpha^2}{4\pi}\leqslant\varepsilon_0 \tag{3.5}$$

with ε_0 small enough. We start by proving that (3.2) has two fixed points different from zero.

We first look for approximate solutions, by imposing in (3.2) the conditions $|Q|$, $|Q_i|\leqslant 2$ and $n\leqslant 2$. Taking into account the symmetry property $v_Q=v_{-Q}$, we obtain a system of two equations:

$$\begin{aligned} v_1' &= \gamma^\varepsilon v_1+av_1v_2-fv_1^3\\ v_2' &= cv_2-bv_1^2 \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} a &= \gamma^2F_2(-1,2)=\gamma^\varepsilon(1-\gamma^{-4(2-\varepsilon)})\\ b &= -\tfrac{1}{2}\gamma^2F_2(1,1)=\tfrac{1}{2}\gamma^{-2(1-\varepsilon)}(1-\gamma^{-4+2\varepsilon})\\ c &= \gamma^{-6+4\varepsilon}\\ f &= -\tfrac{1}{2}\gamma^2F_3(-1,1,1)=\tfrac{1}{2}\gamma^\varepsilon(1-\gamma^{-4+2\varepsilon})^2 \end{aligned} \tag{3.7}$$

If $\varepsilon>0$, the system (3.7) has two solutions different from zero given by

$$\begin{aligned} \bar v_1^2 &= \frac{(1-c)(\gamma^\varepsilon-1)}{ab+f(1-c)}\\ \bar v_2 &= \frac{-b(\gamma^\varepsilon-1)}{ab+f(1-c)} \end{aligned} \tag{3.8}$$

Furthermore, it is possible to see that, if $\bar v_1^+$ and $\bar v_1^-$ are, respectively, the positive and the negative solution, there is a neighborhood $\mathfrak{S}$ of the origin in $\mathbb{R}^2$, containing $\bar v^+$ and $\bar v^-$, such that $\mathfrak{S}\cap\{v_1>0\}$ and $\mathfrak{S}\cap\{v_1<0\}$ are in the domain of attraction, respectively, of $\bar v^+$ and $\bar v^-$.

In the following we shall consider only the solution with $\bar v_1>0$, but the same considerations could be applied to the other one. We want to prove that there is a fixed point of (2.12) which is *approximately equal* to the function

$$\bar v(\varphi)=2\bar v_1[\cos\varphi-1]+2\bar v_2[\cos(2\varphi)-1] \tag{3.9}$$

We want to apply the contraction mapping principle; hence we need to define a suitable Banach space and find a **T**-invariant subset, containing (3.9), on which **T** is a contraction with respect to a suitable metric.

Let us consider the functions of $\mathfrak{C}_\alpha$, which are analytic and bounded in a symmetric strip along the real axis of width $2b$ such that

$$\delta \equiv e^{-\alpha\bar{b}} \equiv \bar{a}\varepsilon^{1/2} \leqslant \bar{\delta} < 1 \tag{3.10}$$

These functions form a Banach space $\mathfrak{B}$, if we define the norm in the following way:

$$\|v\| = \sup_{Q \geqslant 1} \delta^{-Q} |v_Q| \tag{3.11}$$

Let $\mathfrak{B}_d \subset \mathfrak{B}$ be the sphere of radius d with center at the origin. We want to choose d and $\bar{a}$ [see (3.10)] so that the smaller sphere $\mathfrak{B}_{d/2}$ contains the function $\bar{v}$ [see (3.9)]. From (3.8) it follows that this is possible if

$$\begin{gathered} 2\left[\frac{1-c}{ab+f(1-c)}\,\frac{\gamma^\varepsilon - 1}{\varepsilon}\right]^{1/2} \leqslant d\bar{a} \\ 2\,\frac{b}{ab+f(1-c)}\,\frac{\gamma^\varepsilon - 1}{\varepsilon} \leqslant d\bar{a}^2 \end{gathered} \tag{3.12}$$

The bounds (3.12) and (3.10) can be satisfied for any d if $\bar{a}$ is sufficiently large and ε is sufficiently small, as we shall suppose in the following.

We want now to define a subset $\mathfrak{D} \subset \mathfrak{B}_d$ containing the functions which are *close* to $\bar{v}$. Let us define

$$v_i = \bar{v}_i + r_i, \qquad v_i' = \bar{v}_i + t_i', \qquad i = 1, 2 \tag{3.13}$$

and consider the linear change of coordinates which diagonalizes the linearization of the system (3.6) around its fixed point, that is,

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = S \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.14}$$

where

$$S = \begin{pmatrix} 1 & -a\sigma\bar{v}_1 \\ -2b\sigma\bar{v}_1 & 1 \end{pmatrix} \tag{3.15}$$

with

$$\begin{aligned} \sigma &= \frac{1}{1-c-K(\gamma^\varepsilon - 1)} \\ K &= \frac{4(1-c)}{(1+2f\bar{v}_1^2 - c)\{1+[1-8(1-c)(\gamma^\varepsilon - 1)/(1+2f\bar{v}_1^2 - c)^2]^{1/2}\}} \end{aligned} \tag{3.16}$$

$\mathfrak{D}$ is the set of functions $v \in \mathfrak{B}_d$ such that

$$\begin{aligned} |u_1| &\leqslant \tilde{d}\varepsilon^{1+\eta} \\ |u_2| &\leqslant \tilde{d}\varepsilon^{3/2} \end{aligned} \tag{3.17}$$

where $0 < \eta < 1/2$ and $\tilde{d}$ is any fixed positive constant.

Theorem 3.1. There exist a positive constant d_0 and, for any given d, $\tilde{d}$, and η such that $0 < d \leqslant d_0$ and $0 < \eta < 1/2$, another constant ε_0 so that the set $\mathfrak{D}$ is invariant under the transformation $\mathbf{T}$ if $\varepsilon \leqslant \varepsilon_0$.

Proof. (3.2) and (3.4) imply that, if $v \in \mathfrak{B}_d$ and $Q \geqslant 1$,

$$\begin{aligned} |v'_Q| &\leqslant \gamma^{2-Q^2(2-\varepsilon)}\, d\delta^Q + \sum_{n=2}^{\infty} (Bd\sqrt{\kappa})^n e^{-\kappa Q^2/n} \sum_{Q_1+\cdots+Q_n=Q} \prod_{i=1}^{n} |Q_i|\, \delta^{|Q_i|} \\ &\leqslant \gamma^{2-Q^2(2-\varepsilon)}\, d\delta^Q + \sum_{n=2}^{\infty} (B_1 d\sqrt{\kappa})^n e^{-\kappa Q^2/n} \sum_{\substack{Q_1+\cdots+Q_n=Q \\ |Q_i|\geqslant 1}} \prod_{i=1}^{n} \delta_1^{|Q_i|} \end{aligned} \tag{3.18}$$

where B, B_1 are suitable positive constants and δ_1 is chosen so that

$$\bar{\delta} < \delta_1 < 1 \tag{3.19}$$

We now have to carefully bound the sum

$$I \equiv \sum_{n=2}^{\infty} (B_1 d\sqrt{\kappa})^n e^{-\kappa Q^2/n} \sum_{\substack{Q_1+\cdots+Q_n=Q \\ |Q_i|\geqslant 1}} \prod_{i=1}^{n} \delta_1^{|Q_i|} \tag{3.20}$$

We can write

$$I = I_0 + I_1 \tag{3.21}$$

with

$$I_0 = \sum_{n=2}^{Q} (B_1 d\sqrt{\kappa})^n e^{-\kappa Q^2/n} \delta_1^Q N_n(Q) \tag{3.22}$$

$$I_1 = \sum_{n=2}^{\infty} (B_1 d\sqrt{\kappa})^n e^{-\kappa Q^2/n} \sum_{k=1}^{n-1} \binom{n}{k} \sum_{s=k}^{\infty} \delta_1^{Q+2s} N_k(s)\, N_{n-k}(Q+s) \tag{3.23}$$

where the combinatorial factor $N_k(Q)$ is defined as

$$N_k(Q) = \sum_{\substack{Q_1+\cdots+Q_k=Q \\ Q_i\geqslant 1}} 1 = \begin{cases} \dbinom{Q-1}{k-1} & \text{for} \quad Q \geqslant k \geqslant 1 \\ 0 & \text{for} \quad 1 \leqslant Q < k \end{cases} \tag{3.24}$$

It is easy to see that

$$I_0 \leqslant \kappa Q e^{-\kappa Q}(B_1 d)^2 (1 + B_1 d\sqrt{\kappa})^{Q-2} \delta_1^Q \tag{3.25}$$

In order to bound I_1, we use the inequality

$$\binom{s}{k} \leqslant \frac{s^k}{k!} \leqslant \frac{e^{\rho s}}{\rho^k} \tag{3.26}$$

valid for any positive ρ and we choose ρ so that

$$\delta_2 \equiv \delta_1 e^{\rho} < 1 \tag{3.27}$$

Then we have

$$I_1 \leqslant \delta_2^Q (\delta_1 B_1 d\sqrt{\kappa})^2 \sum_{n=2}^{\infty} \left(\frac{B_1 d\sqrt{\kappa}}{\rho}\right)^{n-2} e^{-\kappa Q^2/n} \frac{(1+\delta_2^2)^n - 1 - \delta_2^{2n}}{\delta_2^2(1-\delta_2^2)} \tag{3.28}$$

If d is sufficiently small, we have also

$$\frac{B_1 d\sqrt{\kappa}}{\rho}(1+\delta_2^2) \leqslant \bar{\delta} < 1 \tag{3.29}$$

which is compatible with (3.10) and (3.12) for ε small enough and $\bar{a}$ large enough. (3.22), (3.28), and (3.29) easily imply that

$$I \leqslant \bar{A}(B_1 d)^2 \delta^Q \tag{3.30}$$

for a suitable positive constant $\bar{A}$, only depending on $\bar{\delta}$ and δ_1.

The inequalities (3.18) and (3.20) imply that, if $Q \geqslant 3$, $|v'_Q| \leqslant d\delta^Q$ provided that

$$\gamma^{2-9(2-\varepsilon)} d + \bar{A}(B_1 d)^2 \leqslant d \tag{3.31}$$

which can be satisfied if d is sufficiently small. Then, in order to prove that $\mathfrak{D}$ is invariant, we still have to check that the condition (3.17) is preserved.

Let us notice that

$$\begin{pmatrix} r'_1 \\ r'_2 \end{pmatrix} = \begin{pmatrix} 1 - 2f\bar{v}_1^2 & a\bar{v}_1 \\ -2b\bar{v}_1 & c \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \tag{3.32}$$

where

$$\begin{aligned} \psi_1 &= a r_1 r_2 - 3f\bar{v}_1 r_1^2 - f r_1^3 \\ \psi_2 &= -b r_1^2 \end{aligned} \tag{3.33}$$

and

$$\tilde{v}_1 = \gamma^2 \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{\substack{Q_1 + \cdots + Q_n = 1 \\ |Q_1| + \cdots + |Q_n| \geqslant 5}} v_{Q_1} \cdots v_{Q_n} F_n(Q_1,..., Q_n)$$

$$\tilde{v}_2 = \gamma^2 \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{\substack{Q_1 + \cdots + Q_n = 2 \\ |Q_1| + \cdots + |Q_n| \geqslant 4}} v_{Q_1} \cdots v_{Q_n} F_n(Q_1,..., Q_n) \tag{3.34}$$

Proceeding as before, it is easy to show that, if d is sufficiently small, then

$$|\tilde{v}_1| \leqslant d\delta^5 = d\bar{a}^5 \varepsilon^{5/2}$$
$$|\tilde{v}_2| \leqslant d\delta^4 = d\bar{a}^4 \varepsilon^2 \tag{3.35}$$

The previous considerations imply that there exists $d_0 > 0$ such that, given $d \leqslant d_0$, (3.10), (3.12), (3.29), (3.31), and (3.35) are satisfied for ε sufficiently small.

By some simple algebra and using (3.14), it is possible to show that

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} \tilde{v}_1 + a\tilde{\sigma}\bar{v}_1 \tilde{v}_2 \\ \tilde{v}_2 + 2b\tilde{\sigma}\bar{v}_1 \tilde{v}_1 \end{pmatrix} \tag{3.36}$$

where

$$\lambda_1 = 1 - K(\gamma^\varepsilon - 1)$$
$$\lambda_2 = \gamma^{-6+4\varepsilon} + K(\gamma^\varepsilon - 1) - 2f\bar{v}_1^2 \tag{3.37}$$

and

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \tilde{\sigma} \begin{pmatrix} ar_1 r_2 - (3f + ab\sigma)\bar{v}_1 r_1^2 - fr_1^3 \\ -br_1^2 + 2b\tilde{\sigma}\bar{v}_1(ar_1 r_2 - 3f\bar{v}_1 r_1^2 - fr_1^3) \end{pmatrix} \tag{3.38}$$

$$\tilde{\sigma} = \frac{\sigma}{1 - 2ab\sigma^2 \bar{v}_1^2} \tag{3.39}$$

By using (3.8), (3.14), (3.35), and (3.36), it is easy to prove that, if $0 < \eta < 1/2$, and ε is sufficiently small, say $\varepsilon \leqslant \varepsilon_0$, then

$$|u_1'| \leqslant \lambda_1 \tilde{d}\varepsilon^{1+\eta} + c_1 \tilde{d}^3 \varepsilon^{5/2+\eta} + c_2 d\bar{a}^5 \varepsilon^{5/2}$$
$$|u_2'| \leqslant \lambda_2 \tilde{d}\varepsilon^{3/2} + c_3 \tilde{d}^3 \varepsilon^{2+2\eta} + c_4 d\bar{a}^5 \varepsilon^2 \tag{3.40}$$

for suitable constants c_i, $i=1,\dots,4$. Then the conditions (3.17) are preserved if

$$\begin{aligned} &\lambda_1 + c_1\, \tilde{d}^2 \varepsilon^{3/2} + c_2 \frac{d}{\tilde{d}}\, \bar{a}^5 \varepsilon^{3/2-\eta} \leqslant 1 \\ &\lambda_2 + c_3\, \tilde{d}^2 \varepsilon^{1/2+2\eta} + c_4 \frac{d}{\tilde{d}}\, \bar{a}^5 \varepsilon^{1/2} \leqslant 1 \end{aligned} \tag{3.41}$$

By looking at (3.37), it is immediate to see that (3.41) can be satisfied, given any $0<\eta<1/2$, if ε is small enough. In order to prove that $\mathbf{T}v \in \mathfrak{D}$, we still have to check that

$$|v_1'| \leqslant d\delta, \qquad |v_2'| \leqslant d\delta^2 \tag{3.42}$$

which is again true for any $\eta<1/2$, if ε is small enough, by (3.13), (3.17), and the fact that $\bar{v} \in \mathfrak{B}_{d/2}$. ∎

We now want to show that $\mathfrak{D}$ contains a fixed point v^* of the transformation $\mathbf{T}$ and that, given any $v \in \mathfrak{D}$, $\|\mathbf{T}^n v - v^*\| \to 0$ as $n \to \infty$.

Given two elements of $\mathfrak{D}$, $v^{(1)}$ and $v^{(2)}$, we define $r_i^{(j)} = v_i^{(j)} - \bar{v}_i$, $j, i = 1, 2$, and $u_i^{(j)}$ as in (3.13) and (3.14); then we define

$$\mathfrak{m}(v^{(1)}, v^{(2)}) = \max\{\delta^{-1} |u_1^{(1)} - u_1^{(2)}|, \delta^{-2} |u_2^{(1)} - u_2^{(2)}|, \sup_{Q\geqslant 3} \delta^{-Q} |v_Q^{(1)} - v_Q^{(2)}|\} \tag{3.43}$$

It is easy to see that

$$\begin{aligned} |v_1^{(1)} - v_1^{(2)}| &= |r_1^{(1)} - r_1^{(2)}| \leqslant c_5\, \delta\, \mathfrak{m}(v^{(1)}, v^{(2)}) \\ |v_2^{(1)} - v_2^{(2)}| &= |r_2^{(1)} - r_2^{(2)}| \leqslant c_5\, \delta^2 \mathfrak{m}(v^{(1)}, v^{(2)}) \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} |u_1^{(1)} - u_1^{(2)}| &\leqslant c_5\, \delta\, \|v^{(1)} - v^{(2)}\| \\ |u_2^{(1)} - u_2^{(2)}| &\leqslant c_5\, \delta^2\, \|v^{(1)} - v^{(2)}\| \end{aligned} \tag{3.45}$$

for some constant c_5.

The inequalities (3.44) and (3.45) imply that $\mathfrak{m}$ and the norm (3.11) generate the same topology. Therefore, in order to prove the existence of a fixed point v^* in $\mathfrak{D}$ and its asymptotic stability, the following theorem is sufficient.

Theorem 3.2. There exist a positive constant $d_1 \leqslant d_0$ and, for any given $d \leqslant d_1$ and $0 < \eta < 1/2$, another constant $\varepsilon_1 \leqslant \varepsilon_0$ such that, for any $\varepsilon \leqslant \varepsilon_1$,

$$\mathfrak{m}(\mathbf{T}v^{(1)}, \mathbf{T}v^{(2)}) \leqslant \nu_\varepsilon \mathfrak{m}(v^{(1)}, v^{(2)}) \tag{3.46}$$

with $\nu_\varepsilon < 1$ and $\nu_\varepsilon \to 1$ as $\varepsilon \to 0$.

Proof. By proceeding as in the proof of Theorem 3.1 and using the identity

$$v_{Q_1}^{(1)} \cdots v_{Q_n}^{(1)} - v_{Q_1}^{(2)} \cdots v_{Q_n}^{(2)} = \sum_{k=1}^{n} v_{Q_1}^{(1)} \cdots v_{Q_{k-1}}^{(1)} [v_{Q_k}^{(1)} - v_{Q_k}^{(2)}] v_{Q_{k+1}}^{(2)} \cdots v_{Q_n}^{(2)} \tag{3.47}$$

and (3.44), it is easy to show that, if $Q \geqslant 3$ and d is sufficiently small, then

$$\delta^{-Q} |(\mathbf{T}v^{(1)})_Q - (\mathbf{T}v^{(2)})_Q| \leqslant [\gamma^{2-9(2-\varepsilon)} + B_2 d]\, \mathfrak{m}(v^{(1)}, v^{(2)}) \tag{3.48}$$

for some constant B_2, depending only on $\bar{\delta}$ and δ_1.

Moreover, by using (3.36), it is easy to show that

$$\begin{aligned} \delta^{-1} |u_1'^{(1)} - u_1'^{(2)}| &\leqslant [\lambda_1 + c_6 \varepsilon^{3/2} + c_6 \varepsilon^2]\, \mathfrak{m}(v^{(1)}, v^{(2)}) \\ \delta^{-2} |u_2'^{(1)} - u_2'^{(2)}| &\leqslant [\lambda_2 + c_6 \varepsilon^{1/2+\eta} + c_6 \varepsilon]\, \mathfrak{m}(v^{(1)}, v^{(2)}) \end{aligned} \tag{3.49}$$

for d small enough, say $d \leqslant d_1 \leqslant d_0$, and some constant c_6 depending on $\tilde{d}$, $\bar{a}$, and d_1.

All the claims of the theorem easily follow. ∎

Theorem 3.2 is not completely satisfactory, since we are interested in the properties of the measure (3.5) with potential $v(\varphi) = \lambda(\cos \varphi - 1) \equiv v^{(\lambda)}(\varphi)$. The properties of the approximate transformation (3.6) [see discussion after (3.8) suggest that $v^{(\lambda)}$ is in the domain of attraction of v^* for λ positive and sufficiently small, and that a similar result holds for $\lambda < 0$; moreover, the computer simulation is in complete agreement with this conjecture. In order to prove this claim rigorously, however, we should investigate more accurately the properties of Eq. (3.36), trying to show that $\mathbf{T}^n v^{(\lambda)} \in \mathfrak{D}$ for any λ sufficiently small, if n is large enough. We think that this is possible, but we did not try to fill in the details.

4. THE CORRELATION FUNCTIONS

Let us suppose that ε, η, and d are chosen so that there is in $\mathfrak{D}$ a fixed point v^* of the transformation $\mathbf{T}$. We want to study the linear operator $\mathbf{L}_v$ [see (2.13)] for $v = v^*$; let us simply call it $\mathbf{L}$.

We shall consider the action of $\mathbf{L}$ on the Hilbert space $\mathfrak{H}$ of the functions periodic of period $T_\alpha = 2\pi/\alpha$ with inner product

$$(G, F) = \frac{1}{T_\alpha} \int_0^{T_\alpha} d\varphi\, q(\varphi)\, G^*(\varphi)\, F(\varphi) \tag{4.1}$$

where

$$q(\varphi) = e^{v(\varphi)} \int P_0(dz)\, e^{v(\varphi+z)} \equiv e^{v(\varphi)} N(\varphi) \tag{4.2}$$

Proposition 4.1. $\mathbf{L}$ is a trace class, positive, self-adjoint operator of norm 1.

Proof. It is very easy to verify that $\mathbf{L}$ is self-adjoint, by using the fact that the measure $P_0(dz)$ is even in z.

Let us now observe that the functions

$$\psi_Q(\varphi) = q(\varphi)^{-1/2} e^{i\alpha Q\varphi}, \qquad Q \in \mathbb{Z} \tag{4.3}$$

are a base of $\mathfrak{H}$ and that we can write

$$e^{v(\varphi)} q(\varphi)^{-1/2} = \sum_Q g_Q e^{i\alpha Q\varphi}, \qquad \sum_Q |g_Q|^2 < \infty \tag{4.4}$$

Then we have

$$\begin{aligned}
\mathrm{Tr}(\mathbf{L}) &= \sum_Q (\psi_Q, \mathbf{L}\psi_Q) \\
&= \sum_Q \int P_0(dz) \frac{1}{T_\alpha} \int_0^{T_\alpha} d\varphi\, [e^{v(\varphi)} \psi_Q^*(\varphi)][e^{v(\varphi+z)} \psi_Q(\varphi+z)] \\
&= \sum_{Q,Q'} |g_{Q'-Q}|^2 \int P_0(dz)\, e^{i\alpha Q'z} = \sum_{Q,Q'} |g_{Q'-Q}|^2 \gamma^{-(\alpha^2/4\pi)Q'^2} \\
&= \left(\sum_Q |g_Q|^2\right)\left(\sum_{Q'} \gamma^{-(\alpha^2/4\pi)Q'^2}\right) < \infty
\end{aligned} \tag{4.5}$$

which proves that $\mathbf{L}$ is trace class.

If $F \in \mathfrak{H}$, then $e^v F \in \mathscr{L}_2$ and therefore we can write

$$e^{v(\varphi)} F(\varphi) = \sum_Q \tilde{f}_Q e^{i\alpha Q\varphi}, \qquad \sum_Q |\tilde{f}_Q|^2 < \infty \tag{4.6}$$

Then, by proceeding as before, we can check that

$$(F, \mathbf{L}F) = \sum_Q |\tilde{f}_Q|^2 \gamma^{-(\alpha^2/4\pi)Q^2} \tag{4.7}$$

which proves that $\mathbf{L}$ is a positive operator. Moreover,

$$
\begin{aligned}
(F, \mathbf{L}F) &= |(F, \mathbf{L}F)| \\
&\leqslant \int P_0(dz) \frac{1}{2T_\alpha} \int_0^{T_\alpha} d\varphi\, e^{v(\varphi)+v(\varphi+z)}[|F(\varphi)|^2 + |F(\varphi+z)|^2] \\
&= \frac{1}{T_\alpha} \int_0^{T_\alpha} d\varphi\, e^{v(\varphi)}\, |F(\varphi)|^2\, N_v(\varphi) = (F, F)
\end{aligned}
\tag{4.8}
$$

Since $\mathbf{L}F = F$, if F is a constant function, $\|\mathbf{L}\| = 1$. ▮

By Proposition 4.1, $\mathbf{L}$ is a positive compact operator; then it has a pure discrete point spectrum with positive eigenvalues, at most finitely degenerate. Furthermore, the subspaces $\mathfrak{H}^+$ and $\mathfrak{H}^-$ of $\mathfrak{H}$, which contain the functions even and odd in φ, respectively, are invariant under the action of $\mathbf{L}$. Let $\mathbf{L}^\pm$ be the restriction of $\mathbf{L}$ to $\mathfrak{H}^\pm$.

Since the constants are eigenfunctions of $\mathbf{L}$ and $\mathbf{L}^+$, for any $\varepsilon \leqslant \varepsilon_1$ there is a simple eigenvalue of $\mathbf{L}^+$:

$$\lambda_0^+ = 1 \tag{4.9}$$

For $\varepsilon = 0$ (and hence $v = 0$) the other eigenvalues of $\mathbf{L}^+$ and $\mathbf{L}^-$ are the same, that is,

$$\lambda_n^\pm(0) = \gamma^{-2n^2}, \qquad n = 1,... \tag{4.10}$$

and they are all simple. By using the properties of v proven in Section 3 and known results about the perturbation theory of compact operators, it is possible to show that the eigenvalues of $\mathbf{L}^\pm$ can be written as suitable functions $\lambda_n^\pm(\varepsilon)$, which are continuous in $\varepsilon = 0$. Hence all the eigenvalues different from λ_0^+ are strictly less than 1.

Let $\{\mu_n\}_{n \geqslant 0}$ be the set of all eigenvalues, ordered so that $\mu_{n+1} \leqslant \mu_n$, and let G_n be the corresponding eigenfunctions, normalized so that G_n is real and

$$(G_n, G_m) = \delta_{nm} \tag{4.11}$$

In particular, $\mu_0 = 1$ and

$$G_0 = \left[\frac{1}{T_\alpha} \int_0^{T_\alpha} d\varphi\, q(\varphi)\right]^{-1/2} \tag{4.12}$$

Moreover, it is possible to show, by the technique used below in the proof of Theorem 4.1, that the G_n are smooth functions.

Let us consider a function $F \in \mathfrak{H}$ such that its expansion in terms of the G_n,

$$F(\varphi) = \sum_{n=0}^{\infty} f_n G_n(\varphi) \tag{4.13}$$

has good convergence properties. Then, by (2.11) we have

$$\begin{aligned}\langle F(\varphi_0) \rangle_v &= \langle (\mathbf{L}^k F)(\varphi_0) \rangle_v \\ &= \sum_n f_n \mu_n^k \langle G_n(\varphi_0) \rangle_v \xrightarrow[k \to \infty]{} f_0 G_0 = (G_0, F) G_0\end{aligned} \tag{4.14}$$

Let us now suppose that we want to calculate the correlation between $F(\varphi_0)$ and $F^*(\varphi_x)$. This problem arises, for example, if we are interested in the two-charge correlation; in this case $F(\varphi) = e^{i\alpha\varphi}$, whose expansion has the needed convergence properties, as is possible to show with some standard calculation, using the smoothness of the functions G_n and the fact that they are small perturbations of the functions $e^{i\alpha Q}$.

If h is the smallest integer so that there exists a $\Delta \in Q_h$ containing both 0 and x, we can write

$$\begin{aligned}\langle F(\varphi_0)\, F^*(\varphi_x) \rangle_v &= \langle |(\mathbf{L}^h F)(\varphi_0)|^2 \rangle_v \\ &= \sum_{nm} f_n f_m \mu_n^h \mu_m^h \langle G_n(\varphi_0)\, G_m(\varphi_0) \rangle_v\end{aligned} \tag{4.15}$$

But, by (4.11) and (4.14)

$$\langle G_n(\varphi_0)\, G_m(\varphi_0) \rangle_v = (G_0, G_n G_m) G_0 = G_0^2 (G_n, G_m) = G_0^2 \delta_{nm} \tag{4.16}$$

Then

$$\begin{aligned}\langle F(\varphi_0)\, F^*(\varphi_x) \rangle_v^T &= \langle F(\varphi_0)\, F^*(\varphi_x) \rangle_v - |\langle F(\varphi_0) \rangle_v|^2 \\ &= G_0^2 \sum_{n=1}^{\infty} \mu_n^{2h}\, |f_n|^2\end{aligned} \tag{4.17}$$

and, as a consequence,

$$\langle F(\varphi_0)\, F^*(\varphi_x) \rangle_v^T \underset{x \to \infty}{\simeq} c\mu_1^{2h} = c d_{0x}^{-\tau} \tag{4.18}$$

with c a suitable constant and

$$\tau = -2 \log_\gamma \mu_1 \tag{4.19}$$

We now want to show that

$$\mu_1 = \gamma^{-2} \tag{4.20}$$

at least for ε small enough. We start by observing that, since $\mathbf{T}v = v$, by (2.12),

$$e^{(1/\gamma^2) v(\varphi)} = \frac{\int P_0(dz)\, e^{v(\varphi + z)}}{\int P_0(dz)\, e^{v(z)}} \tag{4.21}$$

By calculating the φ-derivative of both sides, it is easy to check that

$$\mathbf{L} \frac{dv}{d\varphi} = \gamma^{-2} \frac{dv}{d\varphi} \tag{4.22}$$

Since $dv/d\varphi \in \mathfrak{H}^-$, this implies that

$$\lambda_1^-(\varepsilon) = \gamma^{-2}, \qquad \forall \varepsilon \leqslant \varepsilon_1 \tag{4.23}$$

But μ_1, for ε small enough, is the minimum between $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$; hence, in order to show (4.20) it is sufficient to prove that $\lambda_1^+(\varepsilon)$ is smaller than γ^{-2}.

We notice that (2.13) can be written also in the following way:

$$(\mathbf{L}_v F)(\varphi) = \frac{d}{d\lambda} \log \int P_0(dz) \exp[v(\varphi + z) + \lambda F(\varphi + z)] \bigg|_{\lambda = 0} \tag{4.24}$$

If $F \in \mathfrak{H}$, $F \in \mathscr{L}_2$, then we can write

$$F(\varphi) = \sum_Q f_Q e^{i\alpha Q \varphi} \tag{4.25}$$

Moreover, since $(\mathbf{L}_v F)(\varphi)$ does not change if we add a constant to $v(\varphi)$, we can replace in (4.24) $v(\varphi)$ by the expansion $\sum_{Q \neq 0} v_Q e^{i\alpha Q \varphi}$, whose coefficients are the same as in (3.1). Then we obtain

$$\begin{aligned}(\mathbf{L}_v F)(\varphi) = {} & \sum_Q f_Q \gamma^{-(\alpha^2/4\pi) Q^2} e^{i\alpha Q \varphi} \\ & + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \sum_{Q_1, \ldots, Q_n} v_{Q_1} \cdots v_{Q_{n-1}} f_{Q_n} F_n(Q_1, \ldots, Q_n) \\ & \times \exp\left[i\alpha \left(\sum_{i=1}^{n} Q_i \right) \varphi \right]\end{aligned} \tag{4.26}$$

If $\lambda < 1$, the eigenvalue equation $\mathbf{L}_v F = \lambda F$ is satisfied if $f_0 = 0$ and

$$\lambda f_Q = f_Q \gamma^{-(\alpha^2/4\pi) Q^2} + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \sum_{Q_1 + \cdots + Q_n = Q} v_{Q_1} \cdots v_{Q_{n-1}} f_{Q_n} F_n(Q_1, ..., Q_n) \tag{4.27}$$

for $|Q| \neq 0$.

We are interested in the dependence on ε of the eigenvalue $\lambda_1^+(\varepsilon)$ and of the Fourier coefficients $f_Q(\varepsilon) = f_{-Q}(\varepsilon)$ of the corresponding eigenfunction, which we shall normalize so that

$$f_1(\varepsilon) = f_{-1}(\varepsilon) = 1 \tag{4.28}$$

For $\varepsilon = 0$, we have

$$f_1(0) = f_{-1}(0), \qquad f_Q = 0 \quad \text{if} \quad |Q| \neq 1 \tag{4.29}$$

We shall now rewrite (4.27), for $\lambda = \lambda_1^+(\varepsilon)$ as a fixed-point equation in a suitable Banach space, where the existence of a unique solution will follow from the contraction mapping principle.

Let us define

$$G_Q(F) = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \times \sum_{|Q'| \geq 2} f_{Q'} \sum_{Q_1 + \cdots + Q_{n-1} = Q - Q'} v_{Q_1} \cdots v_{Q_{n-1}} F_n(Q_1, ..., Q_{n-1}, Q') \tag{4.30}$$

Equation (4.27) gives, for $Q = 1$,

$$\lambda_1^+(\varepsilon) = \gamma^{-2} + r(\varepsilon) + G_1(F) \tag{4.31}$$

where

$$r(\varepsilon) = \gamma^{-2}(\gamma^{\varepsilon} - 1) + v_2 F_2(2, -1) + \sum_{n=3}^{\infty} \frac{1}{(n-1)!} \times \sum_{Q = \pm 1} \sum_{Q_1 + \cdots + Q_{n-1} = 1 - Q} v_{Q_1} \cdots v_{Q_{n-1}} F_n(Q_1, ..., Q_{n-1}, Q) \tag{4.32}$$

If $|Q| \geq 2$, we have

$$[\lambda_1^+(\varepsilon) - \gamma^{-(\alpha^2/4\pi) Q^2}] f_Q = h_Q + G_Q(F) \tag{4.33}$$

where

$$h_Q = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \sum_{|Q'|=1} \sum_{Q_1 + \cdots + Q_{n-1} = Q - Q'} v_{Q_1} \cdots v_{Q_{n-1}} F_n(Q_1,..., Q_{n-1}, Q') \tag{4.34}$$

By proceedings as in proof of Theorem 3.1, it is easy to show that there exist constants a_1 and A_1, only depending on $\bar{\delta}$ and δ_1, such that

$$|r(\varepsilon)| \leqslant a_1 \varepsilon \tag{4.35}$$

$$|h_Q| \leqslant A_1 \, d\delta^{|Q|-1} \qquad \text{if} \quad |Q| \geqslant 2 \tag{4.36}$$

Let us now consider the Banach space $\mathfrak{E}$ of the even functions $F(\varphi)$, periodic of period T_α, such that $f_0 = 0$, $f_1 = f_{-1} = 1$, and

$$\|F\| \equiv \sup_{Q \geqslant 2} \delta^{-Q+1} |f_Q| < \infty \tag{4.37}$$

We denote by $\mathfrak{E}_D$ the sphere of radius D and center at the origin and we consider the operator $\mathbf{K}$ from $\mathfrak{E}_D$ to $\mathfrak{H}_+$, defined so that, if $(\mathbf{K}F)(\varphi) = \sum_{Q \neq 0} f'_Q \, e^{i\alpha Q \varphi}$, then

$$\begin{aligned} f'_Q = f'_{-Q} &= \frac{h_Q + G_Q(F)}{\gamma^{-2} + r(\varepsilon) + G_1(F) - \gamma^{-(\alpha^2/2\pi) Q^2}} \qquad \text{if} \quad Q \geqslant 2 \\ f'_1 = f'_{-1} &= 1, \qquad f'_0 = 0 \end{aligned} \tag{4.38}$$

By (4.31) and (4.33), F is a solution of (4.27) for $\lambda = \lambda_1^+(\varepsilon)$, belonging to $\mathfrak{E}_D$, if and only if F is a fixed point of the operator $\mathbf{K}$.

Theorem 4.1. There exist a positive constant $d_2 \leqslant d_1$ and, for any given $d \leqslant d_2$ and $0 < \eta < 1/2$, other constants ε_2, D_0, and D_1 such that $\mathfrak{E}_D$ is invariant under the transformation $\mathbf{K}$ if $\varepsilon \leqslant \varepsilon_2$ and $D_0 \leqslant D \leqslant D_1$; moreover, $\mathbf{K}$ is a contraction as an operator from $\mathfrak{E}_D$ to $\mathfrak{E}_D$.

Proof. By proceeding as in the proof of Theorem 3.1, it is possible to show that, if $Q \geqslant 2$ and Dd is small enough,

$$|f'_Q| \leqslant \frac{A_1 d(1+D)\delta^{Q-1}}{\gamma^{-2} - a_1 \varepsilon - DdA_1 \delta - \gamma^{-\alpha^2/\pi}} \tag{4.39}$$

Therefore $\mathfrak{E}_D$ is invariant under the transformation $\mathbf{K}$ if

$$D \geqslant \frac{A_1 d(1+D)}{\gamma^{-2} - a_1 \varepsilon - DdA_1 \delta - \gamma^{-\alpha^2/\pi}} \tag{4.40}$$

and it is easy to see that there exist d_2, ε_2, D_0, and D_1 such that (4.40) is satisfied if $d \leqslant d_2$, $\varepsilon \leqslant \varepsilon_2$, and $D_0 \leqslant D \leqslant D_1$.

Let us now consider two elements $F_1, F_2 \in \mathfrak{E}_D$ such that, for $Q \geqslant 2$,

$$|f_{1Q} - f_{2Q}| \leqslant \rho \delta^{Q-1} \tag{4.41}$$

It is easy to check that, for any $Q \geqslant 1$,

$$|G_Q(F_1) - G_Q(F_2)| \leqslant \rho d A_1 \delta^{Q-1} \tag{4.42}$$

Moreover, if $Q \geqslant 2$, by (4.38),

$$\begin{aligned} f'_{1Q} - f'_{2Q} = \{ & h_Q[G_1(F_2) - G_1(F_1)] + b_Q[G_Q(F_1) - G_Q(F_2)] \\ & + G_Q(F_1)[G_1(F_2) - G_1(F_1)] + G_1(F_1)[G_Q(F_1) - G_Q(F_2)]\} \\ & \times \{[b_Q + G_1(F_1)][b_Q + G_1(F_2)]\}^{-1} \end{aligned} \tag{4.43}$$

where

$$b_Q = r(\varepsilon) + \gamma^{-2} - \gamma^{-(\alpha^2/4\pi)Q^2} \tag{4.44}$$

Then it is very easy to show that, for d small enough,

$$|f'_{1Q} - f'_{2Q}| \leqslant \bar{\alpha}\rho, \qquad \bar{\alpha} < 1 \tag{4.45}$$

which immediately implies, together with Theorem 3.1, all the claims of this theorem. ▮

From (4.30), (4.31), (4.32), Theorem 4.1, and some simple algebra, it follows that

$$\lambda_1^+(\varepsilon) = \gamma^{-2} - \varepsilon \log \gamma \left(1 + \frac{ab}{ab + f(1-c)}\right) + O(\varepsilon^{3/2}) \tag{4.46}$$

Then, if ε is small enough,

$$\lambda_1^+(\varepsilon) < \lambda_1^-(\varepsilon) \tag{4.47}$$

so that (4.20) is satisfied.

This means that, if $v = v^*$, the integer charge truncated correlations decay as d_{xy}^{-4}. With some more computational effort one could show that this result is true also if v is in the domain of attraction of v^*.

We conclude by two remarks. The first remark, anticipated in the discussion preceding (4.13), is that the technique used in this section can be applied to any eigenvalue of $\mathbf{L}_v$ with similar results. In particular, one can show that, for any given n and ε small enough (how small depends on n), $\lambda_n^{\pm}(\varepsilon) < \lambda_n^{\pm}(0)$.

The second remark is that we could study the fractional charge correlations by using similar arguments, in spite of the fact that the function $e^{i\alpha\varphi}$ must be substituted by $e^{i\alpha\xi\varphi}$, $0 < \xi < 1$, which is not periodic of period T_α. It is only sufficient to observe that, if $\bar{F}(\varphi) = e^{i\alpha\xi\varphi}F(\varphi)$, with $F \in \mathfrak{H}$, then

$$(\mathbf{L}_v \bar{F})(\varphi) = e^{i\alpha\xi\varphi}(\mathbf{L}_v^{(\xi)} F)(\varphi) \tag{4.48}$$

$$(\mathbf{L}_v^{(\xi)} F)(\varphi) = \frac{\int P_0(dz)\, e^{i\alpha\xi\varphi}\, e^{v(\varphi+z)}\, F(\varphi+z)}{\int P_0(dz)\, e^{v(\varphi+z)}} \tag{4.49}$$

and $\mathbf{L}_v^{(\xi)}$ is again a self-adjoint operator from $\mathfrak{H}$ to $\mathfrak{H}$, whose spectrum can be studied in the same way as the spectrum of $\mathbf{L}_v$, yielding the same results reported in ref. 6.

APPENDIX. PROOF OF THE BOUND (3.4)

Let $I = \{1,..., n\}$ be the set of the first n positive integers and, for each $i \in I$, let Q_i be a fixed integer ($Q_i \in \mathbf{Z}$). If z_t is a random Gaussian variable with mean 0 and covariance

$$\mathscr{E}(z_t^2) = t \leqslant 1 \tag{A1}$$

and c is a fixed positive constant, we define

$$F(I, t) = \mathscr{E}^T(e^{i\sqrt{c}Q_1 z_t};...; e^{i\sqrt{c}Q_n z_t}) \tag{A2}$$

where $\mathscr{E}^T$ denotes the truncated expectation with respect to z_t. It is a well-known fact that

$$F(I, t) = \exp\left(-\frac{c}{2}\sum_{i=1}^{n} Q_i^2 t\right) f(I, t) \tag{A3}$$

with

$$f(I, t) = \sum_G \prod_{ij \in G} (e^{-cQ_iQ_jt} - 1) \tag{A4}$$

where G is the family of all connected graphs on n vertices labeled by the elements of I, with bonds denoted by ij ($i, j \in I$).

Using the results of ref. 9, in particular Lemma 3.3 and the recurrence relation for $f(I, t)$ which follows from it (see ref. 9, p. 40) it is very easy to show that

$$F(I, t) = \begin{cases} -\dfrac{c}{2}\displaystyle\int_0^t ds\, e^{-(c/2)Q_I^2(t-s)} \sum_{\varnothing \neq J \subsetneq I} Q_J Q_{I\setminus J} F(J, s)\, F(I\setminus J, s) \\ \qquad \text{if} \quad |I| = n \geqslant 2 \\ e^{-(c/2)Q_I^2 t} \qquad \text{if} \quad |I| = 1 \end{cases} \tag{A5}$$

where we used the definition, for J a subset of I,

$$Q_J = \sum_{i \in J} Q_i \tag{A6}$$

We want now to describe the solution of the recurrence relation (A5). Let us consider, for $n \geqslant 2$, the family Γ_n of all planar binary trees with root r and n endpoints labeled by the elements of I, oriented from the root to the endpoints (see Fig. 1).

We call vertices the root, the endpoints (*e.p.* in the following), and the branch points of the tree; the branch points also will be called nontrivial (*n.t.* in the following) vertices. If v is a vertex different from r, we shall denote by v' the vertex immediately preceding it in the tree and we shall say that $i \in v$ if the *e.p.* with label i follows v; moreover, v_0 will denote the vertex immediately following the root. We define

$$Q_v = \sum_{i \in v} Q_i \tag{A7}$$

Finally, we label each vertex v with a real number s_v such that

$$\begin{gathered} t \geqslant s_{v'} \geqslant s_v \geqslant 0 \\ s_r = t \\ s_v = 0 \qquad \text{if } v \text{ is an } e.p. \end{gathered} \tag{A8}$$

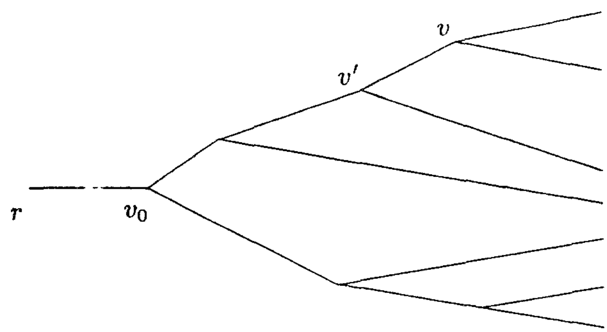

Fig. 1

It is easy to see that, if $|I| \geqslant 2$,

$$F(I, t) = \sum_{\theta \in \Gamma_n} \int \left(\prod_{v\ n.t.} \frac{-ds_v}{2} \right) \chi_\theta W_\theta \tag{A9}$$

where χ_θ is the characteristic function of the set (A9) and

$$\begin{aligned} W_\theta = & \prod_{i=1}^{n} \sqrt{c} Q_i) \left[\prod_{v\ e.p.} \exp\left(-\frac{c}{2} Q_i^2 s_{v'} \right) \right] \\ & \times \left\{ \prod_{\substack{v\ n.t. \\ v \neq v_0}} (\sqrt{c} Q_v) \exp\left[-\frac{c}{2} Q_v^2 (s_{v'} - s_v) \right] \right\} \\ & \times \exp\left[-\frac{c}{2} Q_I^2 (t - s_{v_0}) \right] \end{aligned} \tag{A10}$$

We want to show that

$$|F(I, t)| \leqslant n^{n-1} \prod_{i=1}^{n} (2\sqrt{c}\, |Q_i|) \exp\left(-\frac{c}{4n} Q_I^2 t \right) \tag{A11}$$

The first step in the proof is to get rid of the "bad" factors Q_v in (A10), using the bound

$$\begin{aligned} & \sqrt{c}\, |Q_v| \exp\left[-\frac{c}{2} Q_v^2 (s_{v'} - s_v) \right] \\ & \qquad \leqslant \frac{1}{(s_{v'} - s_v)^{1/2}} \exp\left[-\frac{c}{4} Q_v^2 (s_{v'} - s_v) \right] \end{aligned} \tag{A12}$$

Then, if $|I| \geqslant 2$, we can write, using also that $0 \leqslant t - s_{v_0} \leqslant 1$,

$$|F(I, t)| \leqslant \prod_{i=1}^{n} (\sqrt{c}\, |Q_i|)\, E(I, t) \tag{A13}$$

where

$$E(I, t) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^t \frac{ds}{(t-s)^{1/2}} e^{-(c/4) Q_I^2 (t-s)} \sum_{\varnothing \neq J \subsetneq I} E(J, s)\, E(I \backslash J, s) \\ \qquad \text{if} \quad |I| = n \geqslant 2 \\ e^{-(c/2) Q_I^2 t} \qquad \text{if} \quad |I| = 1 \end{cases} \tag{A14}$$

We now prove, by induction on $n = |I|$, that

$$E(I, t) \leqslant (2n)^{n-1} e^{-(c/4n) Q_I^2 t} \tag{A15}$$

In fact, (A15) is true for $n=1$; moreover, if we suppose that it is true for $1 \leqslant k < n$, we have, using (A14),

$$E(I, t) \leqslant 2^{n-3} \sum_{\varnothing \neq J \subsetneq I} k^{k-1}(n-k)^{n-k-1} G(I, J, t) \tag{A16}$$

where $k=|J|$ and

$$G(I, J, t) = \exp\left(-\frac{c}{4} Q_I^2 t\right) \int_0^t \frac{ds}{(t-s)^{1/2}} \exp\left[\frac{c}{4} s\left(Q_I^2 - \frac{Q_J^2}{k} - \frac{Q_{I\setminus J}^2}{n-k}\right)\right] \tag{A17}$$

If $[Q_I^2 - Q_J^2/k - Q_{I\setminus J}^2/(n-k)] \geqslant 0$, then

$$G(I, J, t) \leqslant \exp\left[-\frac{c}{4} t\left(\frac{Q_J^2}{k} + \frac{Q_{I\setminus J}^2}{n-k}\right)\right] \int_0^t \frac{ds}{(t-s)^{1/2}} \leqslant 2 \exp\left(-\frac{c}{4n} t Q_I^2\right) \tag{A18}$$

where we used the fact that $t \leqslant 1$ and the inequality, valid for two arbitrary real numbers a and b,

$$\frac{a^2}{k} + \frac{b^2}{n-k} \geqslant \frac{1}{n}(a+b)^2 \tag{A19}$$

If $[Q_I^2 - Q_J^2/k - Q_{I\setminus J}^2/(n-k)] \leqslant 0$, then

$$G(I, J, t) \leqslant e^{-(c/4) t Q_I^2} \int_0^t \frac{ds}{(t-s)^{1/2}} \leqslant 2 e^{-(c/4n) t Q_I^2} \tag{A20}$$

(A16), (A18), and (A20) imply that

$$\begin{aligned} E(I, t) &\leqslant 2^{n-2} e^{-(c/4n) Q_I^2 t} \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} \\ &\leqslant (2n)^{n-1} e^{-(c/4n) Q_I^2 t} \end{aligned} \tag{A21}$$

where we used the identity (see ref. 9, Lemma 4.2)

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1) n^{n-2} \tag{A22}$$

Then (A15) is proved; if we insert it into (A13), we obtain the bound (A11).

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REFERENCES

1. J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, *J. Phys. C* **6**:1181–1203 (1973).
2. J. Fröhlich and T. Spencer, The Kosterlitz–Thouless transition in two dimension Abelian spin systems and the Coulomb gas, *Commun. Math. Phys.* **81**:527–602 (1981).
3. D. H. U. Marchetti, A. Klein, and J. Fernando Perez, Power law fall off in the Kosterlitz–Thouless phase of a two-dimensional lattice Coulomb gas, *J. Stat. Phys.* **60**:137 (1990).
4. J. Dimock and T. R. Hurd, A renormalization group analysis of the Kosterlitz–Thouless phase, *Commun. Math. Phys.* **137**:263–287 (1991).
5. G. Benfatto, G. Gallavotti, and Nicolò, The dipole phase in two dimensional hierarchical Coulomb gas: Analyticity and correlations decay, *Commun. Math. Phys.* **106**:277–288 (1986).
6. D. H. U. Marchetti and J. Fernando Perez, The Kosterlitz–Thouless phase transition in two-dimensional hierarchical Coulomb gases, *J. Stat. Phys.* **55**:141–156 (1989).
7. D. Brydges and P. Federbush, Debye screening, *Commun. Math. Phys.* **73**:197–246 (1980).
8. K. Gawedski and A. Kupiainen, Asymptotic freedom beyond perturbation theory, in *Les Houches Summer School* (1984).
9. D. Brydges and T. Kennedy, Mayer expansions and the Hamilton–Jacobi equation, *J. Stat. Phys.* **48**:19–49 (1987).